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পীরিতি all traced in the song No. 379 quoted above. Here the contraction undoubtedly creates doubt about the authorship of Caṇḍidāsa.

THE ARYABHATIYAM

Translation

By P. C. Sengupta

Section I

Dasagitika or the Song of Ten Stanzas.

1. Having made obeisance to the True God, the Supreme Brahman who though One is also Many, Aryabhata speaks here of three things, *Ganita* or Mathematics, *Kala-kriya* or planetary motion determining time, and *Gola* or Spherics.

2. In this section the *Varga* letters are used in square places and the *Avarga* letters in non-square places in denoting numbers; the numbers are begun with *ka*, *ya* (in the units place) is equal to the sum of *ngma* and *ma* (*i.e.*, $5+25=30$); the nine vowels are used in eighteen places both square and non-square and may be used in higher places after the ninth square place (similarly with suitable alterations).

For the sake of expressing all numbers in very compact forms, Aryabhata here invents a new alphabetic notation and in this he succeeds eminently; but the forms of the numbers meant to be committed to memory, are however too difficult to pronounce and the next ten stanzas which he meant to be sung by the learners, appear more or less like a jargon of

words and least suitable for a song. According to the scheme adopted the numbers are expressed as follows :—

The Sanskrit alphabet has twenty-five consonants which are called *Varga* letters and begin with *ka*; it has eight *Avarga* consonants beginning with *ya*. Hence according to the notation, numbers are expressed as in the following scheme :—

Ka	Kha	Ga	Gha	ngma	cha	chha	ja	jha	na
1	2	3	4	5	6	7	8	9	10
T'a	Th'a	da	dha	n'a	Ta	Tha 17 (thought)	Tha 18 (that)	Thha	na
11	12	13	14	15	16			19	20
Pa	Pha	Ba	Bha	Ma					
21	22	23	24	25					
Ya	Ra	La	Wa	Sa	Sa'	Sha	Ha		
30	40	50	60	70	80	90	100		

The first twenty-five letters are called *Varga* letters; the next eight *Avarga* letters denote the corresponding numbers only in first or unit's place, these letters in other non-square places denote only 3, 4, 5, 6, 7, 8 and 9. The square places are of the units, hundreds, ten-thousands, etc., the non-square places, of the tens, thousands, hundred thousands, etc. The vowels take both the Varga and Avarga letters to the different square and non-square places according to their order in the alphabet thus: *a* takes a letter to the first place, *e* to the second, *u* to the third, *ri* to the fourth, *li* to the fifth, *ea* to the sixth, *eai* to the seventh, *o* to the eighth and *ou* to the ninth places. In the scheme of notation this stands like thus :—

Units	Sq. place	*A*
Tens	Non-sq. place	
Hundreds	Sq. place	*E*
Thousands	Non-sq. place	
Ten Thousands	Sq. place	*U*
Hundred Thousands	Non-sq. place	
Millions	Sq. place	*Ri*
Ten Millions	Non-sq. place	
Hundred Millions	Sq. place	*Li*
Thousand Millions	Non-sq. place	
Ten Thousand Millions	Sq. place	*Ea*
Hundred Thousand Millions	Non-sq. place	
Billions	Sq. place	*Eai*
Ten Billions	Non-sq. place	
Hundred Billions	Sq. place	*O*
Thousand Billions	Non-sq. place	
Ten Thousand Billions	Sq. place	*Ou*
Hundred Thousand Billions	Non-sq. place	

Examples.—The combination *khu*=*kh*+u; now kha=2, but the application of the vowel *u* takes it to the third square place and means 20000; again *yu*=*ya*+*u*, ya=3 and is taken to the third non-square place and denotes 300000; *Khyughri*=(*kha*+*ya*) both affected by *u*+*gha* affected by *ri*=20000+300000+40000000=4320000; *chayage ye ngmu su chhlri*=*cha*+*ya*+*ga*×*e*+*ya*×*e*+*ngma*×*u*+*sa*×*u*+(*chha*+*la*)×*ri* (the sign × being used for 'affected by')

=6+30+300+3000+50000+700000+7000000+50000000

=57753336.

3. In a Mahayuga (*i.e.*, 4320000 years) the sun makes 4320000 revolutions, the moon 57753336, the earth 1582237500 eastward, Saturn 146564, Jupiter 364224, Mars 2296824 revolutions respectively; the revolutions of Mercury and Venus are the same as of the sun.

Here Aryabhata speaks of the apparent revolutions of planets round the earth, and definitely states that the earth makes 1582237500 rotations in a Mahayuga. These rotations representing so many sidereal days, we infer that 4320000 years consist of 1577917500 days, and we deduce the following planetary mean motions:—

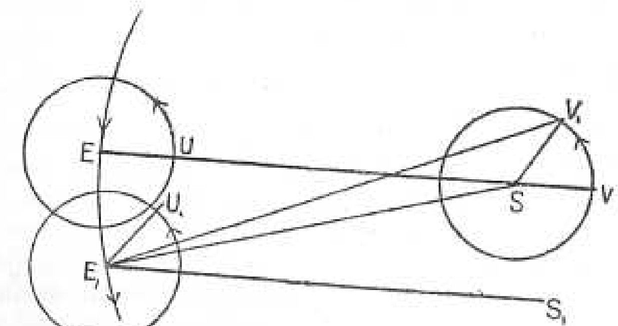
Planet	Sid. Per. according to Aryabhata.	Sid. Per. Modern values.
	Days.	Days.
Moon ...	27·32166852	27·32166140
Sun ...	365·25868005	365·25636042
Mars ...	686·99974	686·9797
Jupiter ...	4332·27216	4332·588
Saturn ...	10765·92819	10759·20

4. The revolutions of the moon's apogee are 488219, of the "*Sighrochchas*" of Venus and Mercury respectively 70222388 and 17937020; the "*Sighrochchas*" of the rest have the same revolutions as the sun. Of the moon's nodes the retrograde revolutions are 232226. All these revolutions start from a Wednesday, sunrise at Lanka—the beginning of the present *Kalpa*, the planets starting from the beginning of the sign *Mesha* (Aries).

From the above stanza we gather that the moon's apogee and nodes perform one complete revolution in 3231·987079 da. and 6794·749511 da. respectively, and the following are obtained from the figures of the two stanzas:—

Length of month.	Aryabhata.	Modern value.
Anomalistic	27·554602 da.	27·554550 da.
Draconitic	27·212425 da.	......
Synodic	29·530582 da.	29·530588 da.

We are next confronted with a term "*Sighrochcha*" and let us try to understand what it means. *Uchcha* is an invisible form of time, a god who seated at his place always drags a planet towards himself, and under the later epicyclic system displaces it parallel to the line joining his place with the earth. *Uchcha* is again of two classes, the first of which is the *Mandochcha*, one producing slow motion, the apogee or the apse of the orbit, the second is the *Sighrochcha* or the one producing quick motion. As the apparent motion of a planet is quickest at the superior conjunction of an inferior planet, and at the conjunction of a superior planet, we shall assume that this *Sighra god* is in directum with the sun at such times and it is further supposed by all Indian astronomers that this god moves round the earth uniformly.



Let ESV be the positions of the earth, sun and Venus at a superior conjunction, let the circles concentric at S be

the orbits of the earth and Venus, the arrow-marks showing the directions of their respective motion; let E_1, V_1 be the positions of the earth and Venus after a certain number of days: let the circles having E and E_1 for the centres be the two positions of the orbit of the *Sighra god*. Looking from E_1 as the mean motion of V is the same as that of the sun, the mean position of V should coincide with S, but for the "attraction" of this *Sighra god*, the planet is seen in the direction of $E_1 V_1$, the mean planet is displaced to V_1; to find where our god is, through E_1 draw $E_1 U_1$ parallel to SV_1, cutting the god's orbit at U_1. At E, the position of the god is at U in the same line with ESV. If $E_1 S_1$ be drawn pa allel to ES, it is evident that the period of the *Sighra god* round the earth is the same as the period of Venus round the sun. Similarly the period of the *Sighra god* of Mercury is the same as its sidereal period. This is borne out as follows:

	Period according to Aryabhata.	Modern Value.
Venus (Sighra)	224·6981 da.	224·7028 da.
Mercury (Sighra)	87·9699 da.	87·9698 da.

A similar treatment with regard to the superior planets leads us to the conclusion that their *Sighra gods* coincide with the mean positions of the sun. Hence their *Sighra gods* have the same period as the sun.

5. There are fourteen *Manus* in a *Kalpa*, the day of *Brahma* and seventy-two *Mahayugas* in a *Manu*. Of the present *Kalpa* six *Manus*, twenty-seven *Mahayugas* of the seventh *Manu* as also the three-fourths of a *Mahayuga* passed before Thursday on which the Great War of the Mahabharata ended.

In this stanza Aryabhata speaks of the time that has elapsed since the beginning of the present *Kalpa* and he first

states his measures of time. Thus a *Kalpa*=fourteen Manus and one Manu=seventy-two Mahayugas. Here he differs from Brahmagupta and the author of Surya-Siddhanta who make one Manu=seventy-one Mahayugas. Again Aryabhata makes the divisions of a Mahayuga all equal, which is not accepted by Brahmagupta and others. He next states the time that elapsed between the beginning of *Kalpa* and the Thursday on which the great war of the Mahabharata ended and which is the beginning of the present *Kaliyuga*. It is 6 Manus + 27 Mahayugas + $\frac{3}{4}$ of Mahayuga, which means 459$\frac{3}{4}$ Mahayugas=725447570625 days. If so many days are counted from a Wednesday, the last day falls on a Wednesday, the next day is Thursday, the beginning of the present *Kaliyuga*.

6. The moon's revolutions multiplied by twelve become the signs, multiplied continuously by 30, 60 and 10, become respectively degrees, minutes and the *Yojanas* of the circumference of the heavens; the earth turns through one minute in a *prana* (=4 seconds); the planets move in circles whose measures are the circumference of the heavens divided by their respective number of revolutions: the orbit of the sun is one-sixtieth of the circle of fixed stars.

In this stanza Aryabhata first speaks of the divisions of a circle into signs, degrees and minutes; he then speaks of the circumference of the heavens and thence of the orbits of the planets. The central idea is that all planets move through the same length of space in a Mahayuga, which length is equivalent to the circumference of the heavens.

The revolutions of the moon 57753336 multiplied by $12\times30\times60\times10$ or 216000, become 12474720576000 which is the measure of the heavens in *Yojanas*.

Here is an attempt to represent planetary motion by equable motion along circles.

7. Eight thousand times the measure (height) of a man equals a *Yojana*, one thousand and fifty *Yojanas* measure the diameter of the earth, the diameters of the sun and the moon are respectively 4410 and 315 *Yojanas*; of the polar mountain

the diameter is 1 *Yojana* and the diameters of Venus, Jupiter, Mercury, Saturn and Mars are respectively $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{20}$, and $\frac{1}{25}$ of the diameter of the moon, when taken at the mean distance of the sun.

The first part defines the unit *Yojana* which is here equal to 8000 times the height of a man, given in the next stanza to be 4 cubits. If we take cubit $=1\frac{1}{2}$ ft. the height of a man is 6 ft. and one *Yojana* $=9\frac{1}{11}$ miles. The diameter of the earth then becomes about 9545 miles and the radius is 4772·5 miles. Uncertainty of the unit, the height of a man renders it impossible for us to judge, if Aryabhata's estimate of the earth is fairly accurate or not. Again if the height of a man is taken at 96 finger-breadths as defined below, we find that 24 finger-breadths make nearly 17 inches, the height of a man becomes 5′-8″ inches and the radius of the earth becomes 4500 miles.

We next come to the diameters of the planets, the sun's diameter is 4410 *Yojanas*; the orbit of the sun is $57753336 \times 216000 \div 4320000$ *Yojanas*; $\therefore$ angular diameter of the sun $=\frac{4410 \times 432000}{57753336} = 32' \; 59{\cdot}28''$, similarly the angular diameter of the moon $=31' \; 30''$

Angular diameters are thus :—

		According to Parameshwara's interpretation.
Sun	32′ 59″·28	32′ 59″·28
Moon	31′ 30″	31′ 30″
Venus	28·29″	6′ 18″
Jupiter	14·14″	3′ 9″
Mercury	9·43″	2′ 6″
Saturn	7·07″	1′ 34″·5
Mars	5·66″	1′ 15″·6

Parameshwara, the Sanskrit Commentator, takes the given diameters of the planets, Venus, etc., to be at the mean distance of the moon, and according to his interpretation their apparent diameters become different. In either case the results from Aryabhata's figures are quite different from all the other writers.

8. The planets' deviation (from the equator) extend up to 24°; the moon's extreme latitude from the ecliptic is $4\frac{1}{2}$ degrees, while those of Saturn, Jupiter, Mars are respectively 2°, 1°, $1\frac{1}{2}$° and of Venus and Mars are 2° each, man is of 96 fingers-breadths or 4 cubits.

The inclinations of planetary orbits can be written in the tabular form thus:—

	Orbital inclination according to Aryabhata.	Modern Values.
Saturn	2°	2° 30′
Jupiter	1°	1° 18′
Venus	2°	3° 24′
Mars	1° 30′	1° 51′
Mercury	2°	7°
Moon	4° 30′	5° 9′

9. The first nodes (ascending) of Mercury, Venus, Mars, Jupiter and Saturn having gone 20°, 60°, 40°, 80°, 100° of longitude are now stationary: the apses of the sun and those planets are at 78°, 210°, 90°, 118°, 180°, 236° of longitude.

10. The peripheries of the *Manda* epicycle (epicycle of apsis) divided by $4\frac{1}{2}$ are, for the moon 7°, and 3°, 8°, 4°, 13°, 7°, 9° respectively for the planets in the order as stated above; similarly the peripheries of the epicycles of quick motion (epicycles of conjunction), are 9°, 16°, 53°, 59°, 30° respectively for Saturn, Jupiter, Mars, Venus and Mercury.

11. The peripheries of the *Manda* epicycle in the 2nd and 4th quadrants of mean anomaly for the one's having

retrograde motion are 5°, 2°, 18°, 8°, 13°; the peripheries of the epicycles of quick motion in the 2nd and 4th quadrants are 8°, 15°, 51°, 57°, 29°; the periphery of the atmosphere is 3375 *Yojanas*.

The figures of the last two stanzas may thus be tabulated :

Peripheries of Epicycles.

	Odd quadrants.	Even quadrants.
Moon	31° 30′	31° 30′
Sun	13° 30′	13° 30′
Mercury	34°...	22° 30′
Venus	17°...	9°...
Mars	58° 30′	61°...
Jupiter	31° 30′	36° 30′
Saturn	40° 30′	58° 30′

Sighra Epicycles (Epicycles of Conjunction).

Saturn	40° 30′	36°
Jupiter	72°	67° 30′
Mars	238° 30′	229° 30′
Venus	265° 30′	256° 30′
Mercury	139° 30′	130° 30′

No comment is necessary just now, we shall take up the subject of Epicycles when we come to discuss planetary motion in the *Kala-kriya*.

12. 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7 are the tabular differences of the half chords expressed in minutes.

Aryabhata here gives the first differences of the sines (*i.e.* perpendiculars) of 24 angles beginning with 225′ and increasing by 225′ in a circle of radius—3438′, so that what

is here spoken of as the sine of an angle is really the sine multiplied by 3438. His figures are exhibited in a tabular form :—

Argument R sin θ,

Angle	Differences	Sines	True Values	True Differences
3° 45′	225	225	224·856	224·856
7° 30′	224	449	448·749	223·893
11° 15′	222	671	670·720	221·971
15′ 0	219	890	889·820	219·100
18° 45′	215	1105	1105·100	215·280
22° 30′	210	1315	1315·666	210·557
26° 15′	205	1520	1520·589	204·923
30° 0′	199	1719	1719·000	198·411
33° 45′	191	1910	1910·050	191·050
37° 30′	183	2093	2092·922	182·872
41° 15′	174	2267	2266·831	173·909
45′ 0′	164	2431	2431·033	164·202
48° 45′	154	2585	2584·825	153·792
52° 30′	143	2728	2727·549	142·724
56° 15′	131	2859	2858·5[illegible]3	131·043
60° 0′	119	2978	2977·495	118·903
63° 45′	106	3084	3083·448	105·953
67° 30′	93	3177	3177·351	93·903
71° 15′	79	3256	3255·546	78·185
75° 0′	65	3321	3320·853	65·307
78° 45′	51	3372	3374·940	51·087
82° 30′	37	3409	3408·588	36·648
86° 15′	22	3431	3430·639	22·051
90° 0′	7	3438	3438·000	7·361

It will be seen that the Table given by Aryabhata is fairly correct, and the values agree with those given in

Surya Siddhanta. We shall take up the method of deriving the Table in the section on *Ganita* (Mathematics).

13. The above ten stanzas of aphorisms meant as a song, embody the various motions of planetary bodies within the limits of this Universe; one who is conversant in these reaches the Supreme Brahman by piercing through the spheres of planets and fixed stars.

The End of Section I named the Dasagitika.

Section II

Ganita.

1. After having done obeisance to the Brahman, the earth, the moon, Mercury, Venus, Mars, Jupiter and Saturn and the fixed stars, Aryabhata here speaks of the knowledge that is worshipped in Kusumpura (*i.e.*, Pataliputra).

2. Units, tens, hundreds, thousands, ten-thousands, hundred thousands, millions, ten-millions, hundred millions, and thousand millions are the notational places; each place is ten times its preceding place.

Here we have the earliest record of the use of notational places (স্থান).

3. The Square is a figure of four equal sides (which has also its diagonal equal); its area is also called a square and the product of two equal quantities is also a square. Product of three equal quantities is called a cube, which is also the name of a twelve-edged figure.

In this stanza Aryabhata does not care to give us any rules for involution as we find in Brahmagupta, Sridhara, Mahavira and Bhaskara.

4. To find the square root always divide from the non-square place by twice the square root (up to the preceding place), if the square of the quotient can be subtracted from the next square place, then this quotient laid down in the next place is the square root.

Example.—Find the square root of 10272025.

Process :—

```
      Sq Sq Sq Sq
      10 27 20 25( 3205
       9
      ----
          12
2×3=6 )12
      ----
           7
     2²=4
      ----
            32     0
2×32=64 )  0   (
      ----
          320
     0²=0,  0
      ------
            3202  5
2×320=640 )3200(
      ------
             25
         5²=25
             --
```

The first step is to mark the square and non-square places which may be done by using dots over the square places as usual. The number up to the highest square place is 10, its square root is 3 which is the first figure of the square root, the remainder 1 with 2 of the non-square place brought down makes up 12, which is divided by twice 3; the quotient is 2, its square must now be subtracted from the figure in the next square place 7 which is possible; hence the next admissible figure of the square root is 2. We have to proceed in the same way till the last figure is exhausted.

Aryabhata here does not clearly state how to find the square root of a non-square number, but it appears likely that he followed a plan like this—

$$\sqrt{3} = \sqrt{\frac{3000000}{1000000}} = \frac{1732}{1000},$$

as is done by Sridhara and Bhaskara and all the later writers. Brahmagupta on this subject gives a definite rule which may be thus summarised :

$$\sqrt{a^2 \pm b} = a \pm \frac{b}{2a \pm \frac{b}{2a}} \quad \text{approximately}$$

According to which

$$\sqrt{3} = \sqrt{2^2 - 1} = 2 - \frac{1}{4 - \frac{1}{4}} = 2 - \frac{4}{15} = 1{\cdot}7333$$

Brahma Siddhanta, XII, 34, 35.

5. To find the cube root divide from the second non-cubic place by three times the square of the cube root, and subtract from the next non-cubic place the square of the quotient multiplied by three times the previous part of the root, and also subtract the cube of the quotient from the next cubic place.

The units' place is the first cubic place, the place of tens is the first non-cubic place and the hundred's place is the second non-cubic place of the first period. Similarly thousand's place is the second cubic place, then again are the first and the second non-cubic places, etc.

Example.—Find the cube root of 34965783.

Process :—

```
                     34965783(327
                     27
                     --
      3×3²=27)        79 2
                      54
                     ----
                     256
      2²×3×3=36        36
                     -----
                     2205
           2³=8         8
                     ------
   2×32²=3072)21977(7
                     21504
                     ------
                       4738
    7²×3×32=4704       4704
                       ----
                        343
          7³=343        343
                        ---
```

∴ the required cube root = 327.

6. The area of a triangle is its *Sarira* (body) and is equal to half the product of the base and the altitude: half the product of that (*i.e.*, the area of the base) and the height is the *Phala* (*i.e.*, the volume) of the six-edged solid (the triangular pyramid).

The second part of the rule is not correct, but strange to say that this rule was blindly followed even by Chaturveda, the Commentator of Brahma Gupta's work ; *cf.* Bhaskara's Goladhyaya, III, 58-61, commentary.

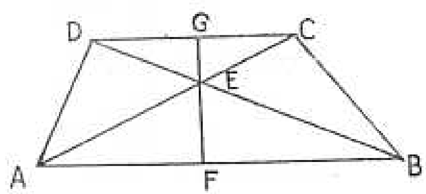
7. Half the circumference multiplied by the radius is undoubtedly the area of the circle; that multiplied by its square root is the volume of the sphere without a residue.

Volume of a sphere is here given as $\pi r^2 \times \sqrt{\pi r^2} = 1{\cdot}47 \times \pi r^3$ which is evidently wrong. The correct value was most probably found by Sridhara; *cf.* Trishatika 56.

8. In a trapezium the distance of the point of intersection of the diagonals, from one of the parallel sides, is the product of the breadth of the trapezium by the corresponding side divided by the sum of the parallel sides. The area of a trapezium is equal to the product of the breadth by half the sum of the parallel sides.

In the trapezium ABCD,

$$EF : EG = AF : GC$$
$$= FB : GD$$
$$= AB : CD$$
$$\therefore EF : FF + EG = AB : AB + CD$$
$$\therefore EF = \frac{GF \times AB}{AB + CD}$$



The second part of the above stanza needs no explanation.

9. Of all figures the length and breadth (average length and average breadth are meant) should be found: the area is their product. Chord of one-sixth of the circumference is equal to radius of the circle.

The second part of the above stanza leads to the problem of squaring the circle or finding the value of π.

10. One hundred and four multiplied by eight with sixty-two thousands is the approximate circumference of a circle whose diameter is twenty thousand.

$$\text{Symbolically } \pi = \frac{62000 + 8 \times 104}{20000}$$

$$= \frac{62832}{20000} = 3{\cdot}1416\,;$$

this is the most accurate value as determined by the ancients.

According to Ahmes the Egyptian

$$\pi = (\tfrac{4}{3})^4 = \tfrac{256}{81} = 3{\cdot}0864.$$

According to Archimedes, $3\frac{1}{7} > \pi > 3\frac{10}{71}$.

According to Apollonius $\pi = 3\frac{17}{120}$,

which is adopted by Ptolemy. A value 3·097, very nearly the same as that of Ahmes, is given in the Sulva-sutras. Some of the later writers Brahmagupta, Mahavira and Sridhara preferred $\sqrt{10}$ as the value of this constant to that given by Aryabhata. They probably wanted to construct a right-angled triangle whose one side is equal to the diameter, and the hypotenuse is equal to the circumference, by doing which the other side of the right-angled triangle becomes nearly equal to three times the diameter. As to the method of Aryabhata I venture to suggest the following which is supported both by Bhaskara and his commentator Ganesha.

Aryabhata most probably began with a regular inscribed hexagon and actually calculated the perimeters of the inscribed regular hexagons of 6, 12, 24, 48, 96, 192, 384 and 768 sides, and equated the perimeter of the regular polygon of 768 sides to the circumference. The results of my calculations by the above method are shown below.

Radius = 10,000.

No. of sides of inscribed regular polygon.	Perimeter.	Perimeter ÷ Diameter.
6	$\sqrt{3600000000}$	3
12	$\sqrt{3858468371}$	3·105828
24	$\sqrt{3925344811}$	3·132629
48	$\sqrt{3942207878}$	3·139350
96	$\sqrt{3946432686}$	3·141032
192	$\sqrt{3947489454}$	3·141452
384	$\sqrt{3947736725}$	3·141550
768	$\sqrt{3947802784}$	3·141575

In the above work, if P be the perimeter of a regular polygon of n sides inscribed in a circle of radius R, the relation used is

$$P' = \sqrt{8n^2R^2 - \sqrt{64n^4R^4 - 16n^2R^2P^2}},$$

where P′ is the perimeter of the regular polygon of $2n$ sides—a relation which is an easy deduction from stanza 17 of this section.

That this was really the method adopted by the Kusumpura School of Mathematicians is corroborated by Bhaskara, Goladhyaya, III, 52, commentary, and also by Ganesha, the commentator on the Lilabati of Bhaskara; *vide* Colebrooke's translation, page 87.

11. Divide a quadrant of the circumference by means of triangles and rectangles, and thus derive the desired 'sines' for any given radius.

When the chord of a given arc is known, half the chord is the sine of half the arc: to find the chord of this half arc, it is necessary to construct a right-angled triangle which will have the chord of the half arc for the hypotenuse; the new chord halved is the sine of half of the half arc; a rectangle has to be drawn for the sine of the complementary arc. Another meaning is that a quadrant is trisected by the inscribed equilateral triangle and bisected by the inscribed square; after these broad divisions the sines of desired arcs may be found. From the way in which the value of π is obtained, it is easy to see that a table of 'sines' was readily derived. *Cf.* the Panchasiddhantika, IV, 1-5; Brahmagupta, XXI, 20-23; Bhaskara's Goladhyaya, V, 1-6.

12. In the table of differences of sines (I, 12) the number by which any figure exceeds its next is equal to the quotient obtained by dividing the sum of the tabular differences from the beginning up to that figure, by the first figure of the table. By these quotients all the remaining figures of the table may be deduced.

If D^n and D^{n+1} be any two consecutive figures of the table, then

$$D_n - D_{n+1} = \frac{\Sigma_1{}^n D_n}{D_1}$$

$$\text{or } D_{n+1} = D_n - \frac{\Sigma_1{}^n D_n}{D_1}, \text{ i.e.,}$$

writing out in full

$$3438 \sin (n+1)\alpha - 3438 \sin n\alpha$$
$$= 3438 \sin n\alpha - 3438 \sin (n-1)\alpha$$
$$- \frac{3438 \sin n\alpha}{225},$$

$$\text{i.e., } \sin (n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin (n-1)\alpha$$
$$- \frac{\sin n\alpha}{225}, \text{ where } \alpha = 225'$$

If the equation be correctly deduced the true divisor comes out to be 234. It is meant to be a rule by which some sines which lie between two well known sines may be derived. The whole table cannot be accurately found even by using the quotient 234. It is further clear that the table was deduced in the right Indian way from the simple rule used in calculating the perimeters of inscribed polygons.

13. The circle is described with a pair of compasses, triangles and quadrilaterals by straight lines; ground is made level with water and the vertical is found with a plumb-line.

14. Add the square of the gnomon to the square of the shadow, the square root of the sum is the radius of the observer's sphere.

This is a very well known sphere in Indian Astronomy used in almost all the standard works, *vide* Surya-Siddhanta III, 22, 23; Brahma-Siddhanta, III, 4, and the Triprasnadhikara of Bhaskara's work. The observer is supposed to be at the end of the shadow. If from the bottom of the gnomon, towards north, a length equal to $h \tan \phi$ where h is the

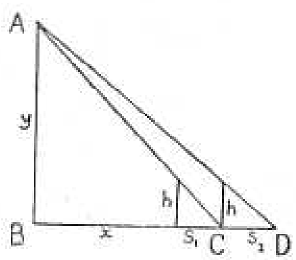
height of the gnomon and ϕ the latitude of the station, be measured and a straight line be drawn through the point thus marked, in the east and west direction, the plane passing through the top of the gnomon and this line, represents any diurnal circle ; and the distance of the end of the shadow from this line measures the sine of the amplitude, where the radius spoken of above is the radius.

15. The distance between the lamp post and the gnomon multiplied by the height of the gnomon and divided by the difference in height between the gnomon and the lamp post is the length of shadow from the bottom of the gnomon.

16. If a lamp post and two gnomons have their bottoms in the same line, the distance between the ends of the two shadows, multiplied by either shadow and divided by the difference of the two shadows, is the base, *i.e.*, the distance between the end of that shadow and the bottom of the lamp post ; that base multiplied by the height of the gnomon and divided by the shadow is the height of the lamp post.

The two gnomons are supposed to be of the same height.

AB represents the lamp post, and h, h the gnomons, S_1, S_2 the two shadows. The rule is evident from the following :—

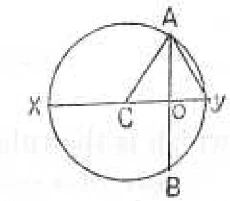


$$\frac{Y}{x}=\frac{h}{s_1},\quad \frac{Y}{x+d}=\frac{h}{s_2};\quad \therefore\ x=\frac{ds_1}{S_2-S_1},\ \text{where } x=BC,\ CD=d$$

17. In a right-angled triangle what is the sum of the squares of the perpendicular and the base, is the square of the hypotenuse. In a circle if a chord divides it into two arcs ; the product of the two arrows of the two arcs is the square of the half chord.

The first half of the stanza gives what is known as the Pythagorian theorem which is given in the Sulva-Sutras ; the theorem was known among all the ancient civilised races, long before the time of Pythagoras. The second half treats of the circle, and is a simple case of Euclid, III, 35.

The chord AB divides the circle ABX into two arcs AYB and AXB, XOY bisects AB at right angles so that XO and OY are the two arrows. Hence $AO^2=XO{\cdot}OY$. If R denotes the radius, then

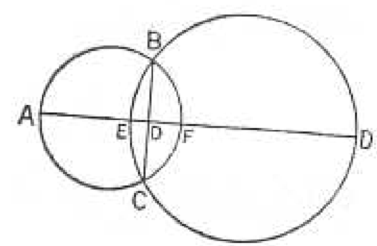


$$OY\,(2R-OY) = AO^2$$
$$\therefore\ 2R{\cdot}OY = AO^2 + OY^2 = AY^2$$
$$\therefore\ AY^2 = 2R\,(R-\sqrt{R^2-AO^2}),$$

from which is readily derived the relation used in calculating the perimeter of a regular polygon of $2n$ sides when the perimeter of the regular polygon of n sides is known.

18. When two circles intersect one another, the sagitta of each is obtained by multiplying the diameter of the other less the *grasha* (the breadth of the common part), by that *grasha* and by dividing the result by the sum of the diameters each diminished by the *grasha*.

Let the circles ABC and BCD intersect at B and C, let AEFD be the line bisecting the common chord BC at right angles at O ; here EF the part of AD representing the breadth of the common part, is called *grasha* or the devoured (eclipsed) portion.



Let $OF=x$ $ED=d'$, and $ED=d=OE=y$, $EF=g$, $AF=d$

$$\therefore\ BO^2 = x\,(d-x) = y\,(d'-y),$$
$$\therefore\ dx-d'y = x^2-y^2 = (x+y)\,(x-y).$$

Again $x + y = g$.

$\therefore dx - d^1y = g\,(x - y)$;

$$\therefore \frac{x}{d^1 - g} = \frac{y}{d - g}$$

$$\text{Hence } x = \frac{g\,(d^1 - g)}{(d - g) + (d^1 - g)}$$

$$y = \frac{g\,(d - g)}{(d - g) + (d^3 - g)}$$

which is the rule.

19. In a series in Arithmetric Progression, the number of terms less one divided by two and multiplied by the common difference and finally increased by the first term is the mean value of a term : that multiplied by the number of terms is the sum of the series up to the desired number of terms ; the same sum is also the sum of the first and last terms multiplied by half the number of terms.

20. When the sum of a series in A. P. is known and also the first term and the common difference, the number of terms is thus obtained : multiply the given sum by eight times the common difference and increase the product by the square of twice the first term diminished by the common difference ; the square root of the result diminished by twice the first term divided by the common difference, increased by unity and finally halved is the number of terms.

If a be the first term and b the common difference, S the sum to n terms, then

$$n = \frac{1}{2}\left(\frac{\sqrt{8bS + (2a - b)^2} - 2a}{b} + 1\right),$$

which readily follows from

$$S = \frac{n}{2}\left\{2a + (n - 1)b\right\}$$

21. In a series consisting of terms which are sums of natural numbers, where the differences increase by unity, the

sum is the continued product divided by six, of the three consecutive numbers of which the first represents the number of terms in the series, or which is the same thing, as the cube of that number increased by unity minus the cube root of the cube and divided by six as before.

Natural numbers are,

$$1, 2, 3, 4, 5, 6, 7 \ldots\ldots$$

Sums of natural numbers,

$$1, 3, 6, 10, 15, 21 \text{ etc}, \ldots \frac{r(r+1)}{2},$$

The stanza says that the sum of the n terms of the second series

$$= \frac{n(n+1)(n+2)}{6} \text{ or } \frac{(n+1)^3 - (n+1)}{6}$$

22. The sum of the squares of the first n natural numbers is one-sixth of continued product of the three numbers of which the first is n, the second is $n+1$, the third is $n+1$ increased by n, the number of terms. The sum of the cubes of n natural numbers is equal to the square of the sum of n natural numbers.

Symbolically the rules are thus expressed :

$$(1) \quad 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(n+1+n)}{6}$$

$$\frac{n(n+1)(2n+1)}{6}$$

$$(2) \quad 1^3 + 2^3 + 3^3 + \cdots n^3 = \left\{1 + 2 + 3 + \ldots + n\right\}^2$$

$$= \left(\frac{n(n+1)}{2}\right)^2.$$

23. To find the product of two numbers, from the square of their sum take away the sum of their squares, half of the result is to be known as their product.

The meaning of this stanza is rather perplexing, if it were simply

$$ab = \frac{1}{2}\left\{(a+b)^2 - a^2 - b^2\right\}.$$

as an algebraical identity there would have been some meaning. But to apply it to find the product of two numbers is to bring in a complexity where none exists.

Example.—Find the product of 2394 and 379 according to the rule: $2394+379=2773$ and we square the numbers by the elegant Indian method, *vide* Colebrooke's Translation of Lilabati.

(1)	(2)	(3)
2773	2394	379
4	4	9
28	12	42
28	36	54
12	16	49
49	9	126
98	54	81
42	24	143641
49	81	
42	72	
9	16	
7689529	5731236	

i.e. $2773^2=7689529$
$2394^2=5731236$
$379^2=\ \ 143641$

$\therefore\ 1814652=2\times2394\times379$

Hence $2394\times379=907326$ Ans.

If this be the new method of multiplication introduced by Aryabhata, we would suggest a modification which would shorten the process by one step.

We have $ab=\frac{1}{4}\left\{(a+b)^2-(a-b)^2\right\}$

Here $a=2394$,

and $b=379$,

$\therefore a+b=2773$

$a-b=2394-379=2015.$

2015
4
0
4
20
001
10
25
4060225

$2773^2=7689529,$
$2015^2=4060225,$

$4\,|\,3629304$

$\therefore\ 2394\times379=907326$ Ans.

It is not unlikely that the above was adopted as an alternative process of multiplication by the Kusumpura school of Indian Mathematicians.

24. Take the square root of the square of the difference increased by four times the product of two unknown quantities: the result increased and decreased by the difference and halved, are the two quantities.

This stanza gives rules for the solution of

$$x-y=a$$
$$xy\ =b$$

25. When the principal is equal to the rate of interest for a given sum, the amount multiplied by the given sum and time is increased by the square of half the given sum; the root of the result diminished by half the given sum and divided by the time is the rate of interest for the given sum.

Let the rate of interest per a rupees $=r$, principal also be r, t denote the time in years or months as the case may be, and A denote the amount, then,

$$A=r+\frac{r^2t}{a}$$

which is a quadratic equation in r;

$$\therefore\ r=\frac{1}{t}\left\{\sqrt{Aat+\left(\frac{a}{2}\right)^2}-\frac{a}{2}\right\}$$

26. In the rule of three, the yield is multiplied by the given number and divided by the argument, the result is the yield of the given number.

27. This is a stanza of obscure import. The first half is totally unintelligible, the second half most probably speaks of reducing fractions to a common denominator.

28. In the inverse operation, the multiplier becomes the divisor and the divisor multiplier; the additive becomes subtractive and the subtractive additive.

Illustration.—What is that number which when multiplied by three, divided by five, increased by six, having the square root taken, diminished by one and finally squared, yields four?

Let x be the required number, then,

$$\left(\sqrt{\frac{3x}{5}+6}-1\right)^2=4,$$

$$\therefore x=\{(\sqrt{4}+1)^2-6\}\times\tfrac{5}{3}.$$

that is in passing from 4 to x, we have first to take the square root, then *add* 1, then square the result, subtract 6, multiply by 5 and divide by 3. This is expressed as if the squaring has become taking the root, the subtractive has become additive, taking square root has become squaring, the additive has become subtractive, the divisor the multiplier and the multiplier divisor.

29. If there are a certain number of unknown quantities and all the possible sums of these quantities less one, be given: then the sum of all the quantities is equal to the sum of all the sums given divided by the number of quantities less one.

If

$$\begin{aligned} b+c+d&=S_1 \\ a+c+d&=S_2 \\ a+b+d&=S_3 \\ a+b+c&=S_4 \end{aligned}$$

then

$$a+b+c+d=\frac{S_1+S_2+S_3+S_4}{4-1}$$

30. If two persons be equally rich, the difference of their wealth in money divided by the difference in things (of equal value) is the price of each thing.

31. When two bodies are moving in opposite directions, their distance divided by the sum of their rates of motion, and when they are moving in the same direction, that distance divided by the difference of their rates, gives the time of their meeting or the time that has elapsed since they last met.

32 and 33. These stanzas are elliptical; the sense has been made out only after comparison with the corresponding stanzas of Brahmagupta.

To find the number which yields known remainders for any two given divisors, divide the divisor yielding the greater remainder by the divisor which yields the lesser remainder and divide mutually the successive remainders; the last remainder is multiplied by such an assumed integer as makes the difference of the original remainders increased by the product, exactly divisible by the last divisor of mutual division; put down the quotients one below the other followed by the assumed number and put down the new quotient last of all. Multiply the lower by the upper and add the last and continue in this operation till only two numbers are left; take the lower of the two numbers, and divide it by the divisor yielding the less remainder; multiply the remainder thus obtained by the divisor yielding the greater remainder and add the greater remainder. The result is the number answering to the two divisors and the two remainders.

Let a number

$$\begin{aligned} N&=A\alpha+K \\ &=B\beta+K^1 \end{aligned}$$

$$\therefore \quad \frac{A\alpha+(K-K^1)}{B}=\beta \quad \cdots \quad \cdots \quad (1)$$

The problem is to find a value of N, when A, B, K and K^1 are known, K being greater than K^1. A is here the divisor yielding the greater remainder and B the divisor yielding the less remainder. The rule directs us to divide A by B, etc.

Suppose

$$
\begin{array}{l}
B\ )\ A\ (\ Q_1 \\
\quad \underline{B\alpha_1} \\
\qquad r_1\ )\ B\ (\ Q_2 \\
\qquad\quad \underline{r_1 Q_2} \\
\qquad\qquad r_2\ )\ r_1\ (\ Q_3 \\
\qquad\qquad\quad \underline{r_2 Q_3} \\
\qquad\qquad\qquad r_3\ )\ r_2\ (\ Q_4 \\
\qquad\qquad\qquad\quad \underline{r_3 Q_4} \\
\qquad\qquad\qquad\qquad r_4\ )\ r_3\ (\ Q_5 \\
\qquad\qquad\qquad\qquad\quad \underline{r_4 Q_5} \\
\qquad\qquad\qquad\qquad\qquad r_5
\end{array}
$$

and also let

$$\frac{r_5 t+c}{r_4}=q \quad \text{where } c=K-K^1$$

Now

$$r_3=r_4Q_5+r_5$$

$$\therefore \quad \frac{r_3 t+c}{r_4}=Q_5 t+\frac{r_5 t+c}{r_4}$$

$$=Q_5 t+q=y_1 \quad \text{suppose}$$

Again

$$r_2=Q_4 r_3+r_4$$

$$\therefore \quad \frac{r_2 y_1-c}{r_3}=Q_4 y_1+\frac{r_4 y_1-c}{r_3}=Q_4 y_1+t=y_2 \quad \text{suppose}$$

Again

$$r_1=Q_3 r_2+r_3,$$

$$\therefore \quad \frac{r_1 y_2+c}{r_2}=Q_3 y_2+\frac{r_3 y_2+c}{r_2}=Q_3 y_2+y_1=y_3 \quad \text{suppose}$$

Again

$$B=Q_2 r_1+r_2$$

$$\frac{By_3-c}{r_1}=Q_2 y_3+\frac{r_2 y_3-c}{r_1^1}=Q_2 y_3+y_2=y_4 \quad \text{suppose}$$

Lastly

$$A=Q_1 B+r_1$$

$$\therefore \quad \frac{Ay_4+c}{B}=Q_1 y_4^{11}+\frac{r_1 y_4+c}{B}=Q_1 y_4+y_3=y_5 \quad \text{suppose}$$

Hence

$\alpha=y_4$ and $\beta=y_5$ is a solution of

$$\frac{A\alpha+c}{B}=\beta$$

The value of y_1, y_2, y_3, y_4, y_5 are deduced by the rule thus :—

Q_1^1	$Q_1 y_4+y_5=y_6$
Q_2	$Q_2 y_3+y_2=y_4$
Q_3	$Q_3 y_2+y_1=y_3$
Q_4	$Q_4 y_1+t=y_2$
Q_5	$Q_5 t+q=y_1$
t	
q	

We are to lay down Q_1, Q_2, Q_3, Q_4, Q_5, t and q in a vertical column t is multiplied by Q_5 and q is added to the product, and q is rejected, then $Q_5 t+q$ is multiplied by Q_4 and t is added thus deriving y_2. We are to go on in this way till only two numbers y_4 and y_5 are left.

It is thus seen that when a solution of

$$\frac{r_5 t+c}{r_4}=q$$

is known, the original equation can be solved.

Now

$$\frac{Ay_4+C}{B}=y_5$$

$$\text{or } Ay_4+C=By_5,$$

$$\text{and } A\alpha+C=B\beta;$$

$$\therefore A(\alpha-y_4)=B(\beta-y_5).$$

$$\therefore \quad \frac{\alpha-y_4}{B}=\frac{\beta-y_5}{A},$$

Again

$$y_4=Bp+\alpha_1 \text{ and } y_5=Ap+\beta_1$$

$$\because \quad \frac{\alpha-\alpha_1}{B}=\frac{\beta-\beta_1}{A}$$

or α_1, the least positive value of α is obtained as the remainder of y_4 divided by B_1 which is the divisor yielding the less remainder.

Hence $N=A\alpha_1+k$ according to the rule.

It is to be noted that t, the assumed integer, is to be selected after an odd number of partial quotients have been obtained, and that y_1 the lower of the two numbers left, is reduced by dividing it by the divisor yielding the lesser remainder. The remainder after an odd number of partial quotients may even be zero.

Example—

Solve $\frac{45x+7}{29} = y$ in the positive integers.

$\frac{45}{29} = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{3}$, so that the remainder is zero after the partial quotient 3.

If we follow Aryabhata's rule we find for the assumed integer 5, $x=208$, $y=323$, the least positive solution is $x=5$, $y=8$.

Aryabhata's method easily led to the methods of all the subsequent writers. In all of them the rule is to proceed backward starting from the last remainder. Parameswara, the Sanskrit Commentator, has totally failed to explain the above stanzas.

The end of Section II named Ganita.

SECTION III

Kala-kriya.

1. A year consists of twelve months ; a month, 30 days ; a day, 60 Nadris ; a Nadri, sixty Binadrikas.

2. One Binadrika is equal to the time of pronouncing sixty long syllables, and is also equal to six Pranas, *i.e.*, respirations. This is the mode of dividing time ; the division of space is in the same way from a *Bhagana*, a whole circle.

TABLE OF TIME.

10 long syllables	=	1 Prana (respiration).
6 Prana	=	1 Binadri.
60 Binadris	=	1 Nadri.
60 Nadri	=	1 day.
30 days	=	1 month.
12 months	=	1 year.

1 day here = the time that the sun takes to move through one degree round the earth, such days are called *Saura* days.

TABLE OF ANGLES.

60 biliptas	=	1 lipta (minute).
60 liptas	=	1 angsa (degree).
30 angsas	=	1 sign.
12 signs	=	a whole circle.
Again 1 bilipta	=	60 tatparas.

3. The difference in the number of revolutions of any two planets represents the number of their conjunctions in a Mahayuga. The revolutions of the sun and the moon added up and doubled represent the number of their *Vyatipatayogas* of these planets.

When the sun and moon have their declinations equal and of the same name, it is called a *Vyatipatayoga*; if the moon had no latitude, this aspect would take place whenever the sum of the longitudes of the sun and the moon is 180; as the moon is not so, this aspect is a matter for accurate calculation. The sun and the moon should be in opposite courses of motion, *i.e.*, if the sun has a northernly course, the moon must have the south course and *vice-versa*.

4. The revolutions of the *Uchcha* (the Apse or the Sighra) diminished by the revolutions of a planet, represent the revolutions of the true planet in the Epicycle (relative to the line joining the mean planet with the earth). The number of revolutions effected by Jupiter since the beginning of the *Kalpa*, multiplied by 12 are the years of Jupiter from Aswayuja. Aswayuja, etc., are the names of such years.

5. In a Mahayuga the number of the sun's revolutions is the same as the number of years, the number of conjunctions of the moon with the sun are the lunar (Synodic) months, the number of conjunctions of the sun with the earth is the number of civil days; the revolutions of the stars represent the sidereal days.

In a Mahayaga of 4320000 years, the

revolution of the sun	=	4320000
revolutions of the moon	=	57753336 (Section I,3)
∴ Synodic months	=	53433336
Solar months	=	51840000
∴ additive lunar months	=	1593336
rotation of the earth or sidereal days	=	1582237500.
revolutions of the sun	=	4320000.
number of civil days	=	1577917500.
number of lunar days	=	1603000080.
omitted lunar days	=	25082580.

6. The lunar months in excess of the solar months are the *Adhimasas* (or intercalary months); the lunar days diminished by the solar days are the omitted lunar days.

7. The solar year is the year of man, that multiplied by thirty is the year of the (departed) fathers, the year of fathers multiplied by 12 is a year of the gods.

8. One thousand divine years multiplied by twelve is the common cycle of all the planets; one hundred and eight of the cycles or Mahayugas is the day of Brahma, the Creator.

The next stanza is explained in various ways.

9. The first half quarter Yuga is progressive, *i.e.*, the period when longevity, physical strength, etc., of all beings go on increasing; the last half quarter Yuga is retrograde when the reverse takes place; the three middle quarter Yugas are even, the first and the last are called Dusshama periods (*i.e.*, periods of greatest variation), all these periods being determined from the position of the moon's apogee.

On this meaning Parameshwara, the last commentator says, "This meaning has got no bearing on the planetary motion." He then gives two other meanings.

9(*b*). The motion of bodies like the moon, her apogee, etc. (*i.e.*, planetary bodies) go on increasing during the first half of the first quarter Yuga and decrease in the last half of the last quarter Yuga and remain even during the three middle quarter Yugas; the first and the last quarter Yugas are the Dusshama portions. It is necessary to correct the mean positions of planets as the mean motions undergo changes.

9(*c*). This interpretation refers to the changes in the Epicycles of planets and the necessary correction thereto.

10. Now when sixty times sixty years and three quarter Yugas also have passed, twenty increased by three years have elapsed since my birth.

The three quarter Yugas are (1) the Krita, (2) Treta and (3) the Dwapara ages. Aryabhata was 23 years old when 3600 years of the present Kali Yuga had elapsed, *i.e.*, in 3600-3179 or 421 of the Saka era which is equivalent to 599 of the Christian era.

11. The Mahayuga, year, month and day started simultaneously from the light half of Chaitra. Time has neither beginning nor end, is estimated by means of planets and stars in the space.

Aryabhata seems to say that time is unvaried and that its variation appears only through the motion of planetary bodies.

12. In sixty solar years all planets moving in their own orbits, describe a length of path equal to the circumference of the steller sphere, and in a divine Yuga the circumference of the heavens.

13. The moon describes her smallest orbit situated beneath all the other orbits in the shortest time, the slow-goer Saturn describes his largest orbit situated above all in the longest time.

14. The signs of the Zodiac are smaller in smaller orbits, the degrees and minutes also vary in size in the same proportion being the same parts of respective orbits.

15. Beneath the stars are the planets in the order, Saturn, Jupiter, Mars, the sun, Venus, Mercury and the moon; beneath all these is the earth stationed at the centre of space like a tying post for all.

16. These seven beginning with Saturn are in their (increasing) order of quickness and are the lords of the hours; the fourths in order of quickness are the successive lords of the days beginning with the sunrise.

This stanza relates to some astrological terms. The lords of the successive hours are in the order of increasing quickness;—Saturn, Jupiter, Mars, the sun, Venus, Mercury and the moon. Of Saturday the first hour belongs to Saturn, the second to Jupiter, third to Mars, etc.; in this way the first hour of the next day belongs to the sun, and the sun, the fourth in the above order, is the lord of the next day. In this way the days of the week belong to Saturn, the sun, the moon, Mars, Mercury, Jupiter and Venus. This sort of naming the days is quite universal and seems to be associated with the discovery of the seven planets. This explanation of determining the lords of the days of the week by Aryabhata is opposed by both Varahamihira and Brahma Gupta. (*Vide* the Pancha-Siddhantika, XV, 26-29, Brahma-Sphuta-Siddhanta, XI, 12 and 13.) Both of them call this illogical because there cannot be any one day all over the world and they prefer to reckon time with lunar days which are dependent on the longitudes of the sun and the moon. Varaha says, "Ordinary people, as a rule, proceed in their business according to the days of the week as known from tradition." The Surya-Siddhanta explains the meaning of the day in the same way as Aryabhata (Surya-Siddhanta, XII, 78 and 75). This sort of the naming of the days of the week, as also the signs of the Zodiac, is to be ascribed to the Babylonians (see History of Astrology in the Encyclopaedia Britannica) from whom most probably both the Indians and the Greeks borrowed their astronomical as astrological knowledge.

17. All planets move in circles eccentric to the orbits which are concentric with the earth at the rates of their mean motion, in the directions of the signs of the Zodiac (eastward) from their apses and in the opposite direction from their *Sighrochchas.*

Sigrochcha is the mean place of the sun in the case of the superior planets; and in the case of the inferior planets, it is in the line through the earth drawn parallel to the line joining the sun to the mean heliocentric position of the inferior planet. This *Sighra god* always moves faster than the mean planet. The business of an *Uchcha* (the apse or the *Sighra god*) is to drag a planet always towards itself by means of strings *made of air*, thus producing inequalities in planetary motion. This most probably was the older idea of planetary motion in ancient India, till it was replaced by the ideas of motion in eccentric circles or in epicycles and deferents. Both these ideas are woven into an incongruous whole in the modern Surya-Siddhanta; see II, 1-5, 9-10, 52; for the epicyclic construction, see II, 34-51, 53-57; see also Dasagitika, 4, Explanations of *Sighra.*

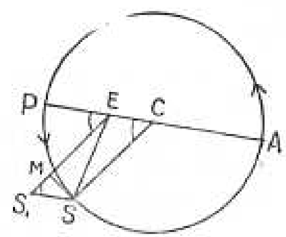
18. The eccentric circles of planets are equal to their concentrics, and the centre of the eccentric is removed from the centre of the earth.

19. The distance between the centre of the earth and the centre of the eccentric is equal to the radius of the planet's

epicycle; on the circumference of the eccentric the planet undoubtedly moves with the mean motion.

These three stanzas describe the eccentric circle idea of the solar system. The central idea was that there was no doubt that the planets described circles round the earth; if the motion were unequal, it was due to the fact that the centres of such circles did not coincide with the centre of the earth.

Let the eccentric circle A P S be the orbit of the sun relative to the earth, E the centre of the earth and C the centre of the eccentric, P and A are respectively the perigee and apogee in the same line with E and C. S moves uniformly the circle, but seen from E motion along appears uneven. Through E and



S let ES_1 and SS_1 be drawn parallel to CS and CE respectively. From S draw SM perpendicular to SE. Let ECS, the mean anomaly, be denoted by θ_1 then the angle SES is the equation of the centre and let it be denoted by ϕ.

Now $\tan\phi=\frac{p \text{ Sin } \theta}{R-p \text{ Cos } \theta}$, where $p=EC$ and $R=CS$,

$$\therefore\ \phi=\frac{p}{R}\text{ Sin }\theta+\frac{p^2}{2R^2}\text{ Sin }2\theta+\frac{p^3}{3R^3}\text{ Sin }3\theta+\ \ldots$$

which is the expression for the equation of the centre in eccentric circle, but the expression for it in elliptic motion is

$$=2e\text{ Sin }\theta+\frac{5}{4}e^2\text{ Sin }2\theta+\frac{e^3}{12}(13\text{ Sin }3\theta-3\text{ Sin }\theta)$$

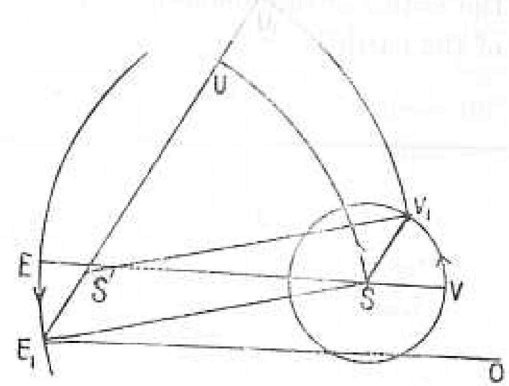
$$\therefore\ \text{if }\frac{p}{R}=2e,\ \frac{p^2}{2R^2}=2e^2$$

which is greater than $\frac{5}{4}e^2$; hence even with a suitable value of $\frac{p}{R}$, the expression is wrong in the co-efficient of the second

power of eccentricity. Hence the laying of emphasis on the planets moving in eccentric circle is due to imperfect methods of observation that the astronomers of those times had to adopt.

Now to come to the other planets, if we suppose them to describe circles uniformly about the sun at the centre, the relative motion round the earth can be represented by eccentric circles and equally well with the help of epicycles and deferents.

In the figure let S represent the sun and the two concentric circles represent the orbits of (1) an inferior planet and (2) of the earth; let E, S, V be the position of the bodies initially so that V is at a superior conjunction and let E_1, S, V_1 be their positions after a certain number of days. Now complete the parallelogram E_1 S V_1 S^1; with



centres E and S^1 draw two circles each having the same radius E S or E_1 S and produce E_1 S^1 cutting these circles in U and U_1. The points U and U_1 are the *Sighra* apogees of V. Through E_1 draw E_1 O parallel to ES. The mean motion of an inferior planet round the earth being the same as that of the sun, looking from E_1, S is the mean position of V and the motion of the mean planet is the general direction of planetary motion. Again as the angular motion of V is greater than that of the earth, the angle SE_1 U goes on increasing, arc U_1 V_1 being always equal to US, looking from U_1, V_1 appears to move in the opposite direction.

In the above figure if the circle having E_1 as the centre, is the concentric the circle having S^1 for the centre is the

eccentric. Again if the circle having E_1 for the centre is called the deferent the real orbit of V is the epicycle. In the case of a superior planet also, the drawing of the similar parallelogram easily leads to the eccentric circle and the epicycle. Further the epicyclic idea is the more natural in the case of inferior planets and the revolving eccentric circle idea, the more natural in the case of superior planets. It is seen that the radius of the epicycle bears the same ratio to the radius of the deferent as the mean distance of an inferior planet bears to the mean distance of the earth; also the radius of the epicycle of a superior planet has the same ratio to the radius of the deferent of the planet as the mean distance of the earth bears to the mean distance of the superior planet.

Mean Distances of Planets.

	Deduced from Aryabhata's Epicycles	Deduced from Ptolemy's Epicycles	Modern Values
Mercury	·375	·375	·3871
Venus	·7250	·7194	·7233
Earth	1·0000	1·0000	1·0000
Mars	1,5385	1·5190	1·5237
Jupiter	5·1613	5·2174	5·2028
Saturn	9·4183	9·2307	9·3389

20. The planet which moves quickest at the *Uchcha* (the highest point, the furthest distance) moves in the reverse direction in its eccentric; while the one which moves slowest at the *Uchcha* moves in the direction of the signs in its eccentric.

21. The *Manda* epicycles move in the direction of planetary motion from the apse, and the *Sighra* epicycles move in the opposite direction from the *Sighrochcha*; the mean planet is at the centre of the epicycle which is on the *Kaksha* (the concentric or deferent).

The above is the only stanza in which planetary motion has been explained on the epicyclic hypothesis.

22. The sines and versed sines for calculating the equation are to be taken negative, positive, positive and negative from the apse (apogee) and they are to be taken in the reverse order from the *Sighrochchas*.

The equation of apsis is negative from the apogee to the perigee and is positive from the perigee to the apogee. The expression for the equation of apsis

$$= -\frac{\text{Periphery of the epicycle} \times \text{R sin (mean anomaly)}}{360}$$

Where R=3438.

Aryabhata determines the 'Sine' of an arc always by the angle *passed over* in any quadrant. When the arc is less than 90°, its 'Sine' is obtained with the help of table given in the Dasagitika, 12; when the arc is greater than 90° but less than 180°, the versed sine of the excess of the arc over 90° subtracted from the radius gives him the desired sine thus :—

$$R \sin(90^\circ+\theta) = R - R \text{ vers } \theta$$

Similarly

$$R \sin(180^\circ+\theta) = -R \sin \theta$$

and

$$R \sin(270^\circ+\theta) = -(R - R \text{ vess } \theta)$$
$$= -R + R \text{ vers } \theta.$$

Thus in the expression for the equation of the centre where the signs of these functions are changed, the sine in the first quadrant is negative, the versed sine in the second is positive, the sine in the third is positive, and the 'versed sine' in the 4th is negative.

The expression for the *Sighra* equation being positive, the sines and the versed sines have the opposite signs.

23. In the case of Saturn, Jupiter and Mars, in the first operation half the equation of apsis is applied negatively or positively; to the thus corrected planet is applied half the corresponding *Sighra* equation (equation of conjunction); to then corrected mean planet is applied the whole of the corresponding equation of apsis and finally the apparent planet is

obtained by applying the whole corresponding *Sighra* equation to the last corrected mean planet.

The above rule is similar to the one given in the Surya-Siddhanta, II.

24. In the case of Venus and Mercury, half the *Sighra* equation is to be applied negatively and positively to the apse, then apply the equation of apsis due to the corrected apse, to the mean planet, and we have the corrected mean planet: to this must be applied the two more equations due to apsis and conjunction.

25. A stanza of obscure import.

The End of Section III, called the Kala-kriya.

Section IV

Gola or Spherics.

1. A half of the ecliptic from the first point of Aries to the last point of Virgo lies on the north of the equator while the other half, from the first point of Libra to the end of Pisces lies on the south.

2. The nodes of the 'star planets'—the five planets, Mercury, Venus, Mars, Jupiter and Saturn—and also of the moon, move unceasingly along the ecliptic, so too does the sun: in the same circle undoubtedly moves the shadow of the earth at the distance of half a circle from the sun.

3. The moon moves north and south of the ecliptic from her nodes, Jupiter, Mars and Saturn also move in the same way from their nodes; so also move Mercury and Venus by the positions of their *Sighrochchas*.

The latitude of the moon depends on the arc between her apparent position and the position of a node. The heliocentric latitudes of other planets also depend on the difference in the heliocentric longitude of the planet and of the node. In the case of a superior planet the Surya-Siddhanta applies the annual parallax to the node, in the same way as it is applied to the planet's heliocentric longitude to convert it to geocentric longitude (the application of the last *Sighra phala*), thus an approximate difference in the heliocentric longitude of the planet and of the node is obtained, Surya-Siddhanta II, 55 56. In the case of an inferior planet the same work applies to the equation of apsis in the opposite way to the node; an approximation to the suitable arc is obtained by taking the difference of the longitudes of the corresponding *Sighra god* and the corrected node, the longitude of the *Sighra god* being the same as the mean heliocentric longitude of the inferior planet. Here Aryabhata also suggests the same thing. In the face of these rules it would be rash to assert that the ancient Indian

astronomers blindly adhered to the geocentric idea of planetary motion. The rules suggest that they really recognised, if not exactly, the heliocentric, at least the Tycho Brahean idea of the planetary system.

4. When the moon's latitude is nil, she becomes heliacally visible when she is at the distance of twelve degrees from the sun, Venus at nine degrees and the others of smaller diameter in the ascending order, at distances increasing by two degrees.

The above can be tabulated as follows:—

Planet	Distance of visibility from the sun
Moon	12°
Venus	9°
Jupiter	11°
Mercury	13°
Saturn	15°
Mars	17°

The increasing order of smallness is given in section I. 7. The visibility of a planet or heavenly body by the naked eye, depends on (1) the brightness of the body, and (2) the corresponding depth of the sun below the horizon a little before sunrise or a little after sunset, according to the position of the body west or east of the sun. The brightness depends on (*a*) the phase of the body and (*b*) the diameter, (*c*) the distance of the body from the earth. The rule of the text is only an approximation.

5. Halves of the spherical bodies of the earth, planets and the stars are darkened by their shadows; the other halves which are presented to the sun shine according to their sizes

6. The earth which is spherical in every way, is inside the circular frame of the constellations, surrounded by the orbits of planets and is situated at the centre of the space. The body of the earth is made of earth, water, fire and air.

7. Just as the knot of the *Kadamba* flower is covered on all sides by flowerets, so is covered the sphere of the earth by all living beings either of land or water.

8. In a day of Brahma the Creator, the earth increases by one *Yojana* radially and in his night which is equal to his day, the earth undergoing a slow contraction, has an equal diminution.

9. Just as a man going east by a boat sees a stationary thing as if it is going west, so do the fixed stars appear to be going straight westward at Lanka.

10. The frame of the constellations with all the planets daily move straight westward at Lanka being thrown that way by the wind called *Prabaha* for producing their daily risings and settings.

Aryabhata is apparently contradicting his statement in the previous stanza; he seems here unable to construct a new idea of motion in place of the old equable motion of heavenly bodies which he states in sec. I, 6. It is further not impossible that this stanza has undergone drastic emendations from his commentators.

11. The *Meru*, *i.e.*, the north polar mountain is only one *Yojana* in height, is an offshoot of the Himalayas in the centre of the celestial garden *Nandana* and is made of jewels and circular in every way.

12. The *Meru* is the heaven at the centre of the land hemisphere, the hell and the *Badabamukha* are in the water hemisphere; the immortals, the dwellers of the heaven and the mortals, the dwellers of the hell, each always consider the other as living underneath.

13. What is sunrise at Lanka is sunset at Siddhapura, is noon at Yavakoti and is midnight at Romaka.

It is now impossible to identify the four cities, Lanka, Siddhapura, Yavakoti and Romaka. They are probably four hypothetical cities on the equator dividing it into 4 equal parts.

14. Lanka is midway between the land and water hemispheres, is at one-fourth of the circumference of the earth (from the pole) and Ujjain is at one-fourth of that quadrant straight towards the north.

The latitude of Avanti is thus $22\frac{1}{2}$ degrees north, but according to Varahamihira and Brahma Gupta it is 24°; Bhaskara however is in agreement with Aryabhata.

15. A half of the circle of constellations diminished by half the diameter of the earth, is visible by the observer, the other half increased by the semi-diameter of the earth is hidden by the earth.

According to Aryabhata the circumference of the heavens of the visible universe

$$=57753336 \times 30 \times 60 \times 10 \times 12 \quad \text{Yojanas.}$$

$\therefore$ the orbit of the sun

$$=\frac{57753336 \times 30 \times 60 \times 10 \times 12}{4320000} \quad \text{Yojanas.}$$

$\therefore$ the circle of constellation

$$=14438334 \times 12 \quad \text{Yojanas.}$$

The diameter of the earth $=1050$ *Yojanas*; $\therefore$ horizontal parallax of any point of the ecliptic

$$=\frac{472500}{7219167} \times 60 = \frac{28350000''}{7219167} = 3''\cdot626.$$

16. The gods living in the north pole see the half of the circle of constellations moving towards the left, whereas the departed in the south pole see the other half moving towards the right.

17. The gods see the sun above the horizon for half of a solar year and the departed souls in the south pole see him in the same way: the fathers living on (the other side of) the moon see the sun for the half of a lunar month and here men for half a civil day.

Here is a recognition of the fact that the same face of the moon is presented to the earth, or that her periods of rotation and revolution are the same.

18. One circle of the observer passes through the east, west, zenith and nadir which is the prime vertical; another circle passes through the north, south, zenith and nadir and is the north south circle, another circle is the one bisecting the first two circles which is the *Kshitija* or the horizon where heavenly bodies rise and set.

Here we have the rational horizon and not the apparent horizon.

19. The circle which passes through the east and west points and also the two poles of the heavens which are respectively raised and lowered from the horizon, is called *Unmandala*, the sixth hour circle, which shows the decrease and increase of the day and night.

20. The intersection of (1) the line joining east and west points, (2) the line joining the zenith and the nadir, and (3) the north and south line, is at the place where the observer is.

21. A circle which passes through the zenith, nadir, and a planet is a *Dringmandala*, a vertical circle; the *Drikshepa-mandala* is the vertical through the nonagesimal passing through a point of the horizon which is 90° west of the ascending point.

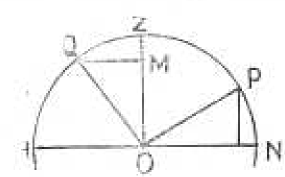
The second half of the stanza cannot be properly interpreted.

22. A light globe made of wood, perfectly spherical and of even weight all round should be turned round by the observer in the unit of time through his intelligence by means of mercury, oil and water.

Here a form of clock is suggested, and seems to be similar to a hydraulic torniquet with a regular flow of water in it, oil being used to diminish friction between parts.

23. In a half of the visible hemisphere a half circle (the meridian circle) is to be marked by means of sines of the latitude and co-latitude; the sine of the latitude is the sine of the meridian zenith distance of the sun at an equinox and is the base, and the sine of the co-latitude is the perpendicular.

Let NPZQH be the semi meridian above the horizon where N is the north point, P the celestial pole, Z the zenith of the observer, O, Q the intersection of the meridian and the celestial equator and H the south point. The stanza says



PR the perpendicular from P on NH $=$ QM the perpendicur from Q on OZ; QM is the sine of the meridian zenith distance of the sun on an equinox. OM the perpendicular when QM is the base of a right angled triangle, is the sine of the colatitude.

24. Subtract the square of the sine of any assumed declination from the square of the radius, the square root of the result is the radius of the corresponding diurnal circle north or south of the equator.

25. Take the sine of an assumed arc (of the ecliptic) from the first point of *Mesha* (Aries), multiply by it the day radius of the extreme declination of the sun and divide by the radius of the corresponding diurnal circle; the result is the sine of the time that the arc will take to rise above the horizon at Lanka.

Let ω be the extreme declination of the sun; then

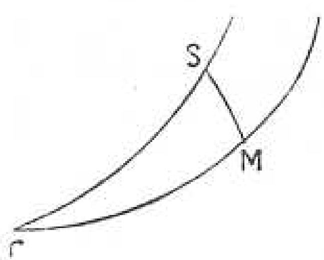
$$\text{Sine of } \omega = \text{R} \sin \omega$$

$$\text{Day radius of } \omega = \sqrt{\text{R}^2 - \text{R}^2 \sin^2 \omega} = \text{R} \cos \omega.$$

Let l be the arc of the ecliptic measured from the vernal equinox and R.A. the corresponding right ascension; the rule says,

$$\text{R Sin R.A} = \frac{\text{R} \cos \omega \times \text{R} \sin l}{\text{R} \cos \delta},$$

where δ is the declination corresponding to the longitude l. This equation readily follows from the spherical triangle γSM where γS is an arc of the ecliptic, SM the declination of S, γM the right ascension corresponding to γS, the angle MγS is ω. We get



$$\tan \gamma \text{M} = \cos \omega \tan \gamma \text{s}. \quad \dots (1)$$

$$\text{and} \quad \cos \gamma \text{M} = \frac{\cos \gamma \text{S}}{\cos \text{SM}} \quad \dots (2)$$

$$\therefore \text{Sin } \gamma \text{M} = \frac{\cos \omega \sin \gamma \text{S}}{\cos \text{S M}}$$

For the Indian method of arriving at the rule, see the Siddhanta-Siromani, Goladhyaya, VII, 43 and also see the Papers on Hindu Mathematics and Astronomy by the translator, pages 48-49.

26. Multiply the sine of the latitude by the sine of any assumed declination and divide by the sine of the co-latitude, the result is the sine in the diurnal circle of the increase and decrease of the day and night and is called the *Kshitijya* or the earth sine.

Here the *Kujya* or the *Kshitijya*

$$= \frac{\text{R} \sin \phi \times \text{R} \sin \delta}{\text{R} \cos \phi}, \ \phi \text{ denoting}$$

the latitude of the observer, the *Kujya* is a sine (perpendicular) in the diurnal circle whose radius is R cos δ; if the *Kuiya* is reduced to the circle of radius R, it becomes the *Charajya* or the sine of the ascensional difference:

$$\therefore \text{Charajya} = \frac{\text{Kujya} \times \text{R}}{\text{R} \cos \delta}$$

$$= \frac{\text{R Sin } \phi \times \text{R Sin } \delta \times \text{R}}{\text{R} \cos \phi \times \text{R} \cos \delta}$$

The *Charajya* is the sine of the time by which half a day differs from 6 hours. For the Indian method of deduction see my Papers, pages 53 and 54.

27. The first and the last quadrants of the ecliptic certainly rise in a quarter of a (sidereal) day diminished by the ascensional difference; the other two quadrants, in the same time increased by the same ascensional difference: the order in which the ascensional difference is applied being negative, positive, positive and negative in the four quadrants.

Aryabhata here indicates the method by which the lengths of time that the signs of the zodiac take to rise at any latitude. For detailed rules we have to go to the subsequent writers Varahamihira, Brahmagupta and others.

28. Take the sine in the diurnal circle of the time from the horizon, multiply it by the sine of co-latitude and divide by the radius; the result the sine of the sun's altitude corresponding to the part of the day elapsed or to the remainder of the day.

This is a very rough rule for finding the sun's altitude at any time of the day. The correct rules for finding this are given by Varahamihira, Brahmagupta and all the subsequent writers. *Cf.* Pancha-Siddhantika, IV, 45-47, Brahma-Siddhanta, 38-40.

The problem of finding time by altitude was first solved by Indian Astronomers, *vide* my paper "Time by Altitude in Indian Astronomy" in the Bulletin of the Calcutta Mathematical Society, Vol. XVIII, No. I. For the Indian method see my "Papers on Hindu Mathematics and Astronomy," pages 57-59.

29. Multiply the sine of the sun's altitude by the sine of the latitude and divide by the sine of the co-latitude: the result is called the sun's *Sankvagram* (*Sankutala* of later writers), which lies south of line of intersection of the diurnal circle and the horizon.

If from the position in the armillary sphere of the sun at any time of the day, a perpendicular be drawn on the horizon, the distance of the foot of this perpendicular from the line of intersection of the diurnal circle and the horizon is called the *Sankvagram* or the *Sankutalam*. This line is used in Indian astronomy in finding the equation connecting the altitude and the azimuth of the sun.

30. Multiply the sine of the sun's extreme declination by the sine of the sun's longitude and then divide by the sine of the co-latitude, the result is the *Agra* or the sine of the sun's amplitude in the eastern or western horizon.

Symbolically it means :—

$$\text{R sin (amplitude)} = \frac{\text{R sin } \omega \times \text{R sin } l}{\text{R cos } \phi}$$

where ω is the obliquity of the ecliptic, l the sun's longitude, and ϕ the latitude of the station. For the Indian method of arriving at the equation see my "Papers," pages 54 and 55.

31. When this *Agra* is less than the sine of the latitude and is north of the equator, multiply it by the sine of co-latitude and divide by the sine of the latitude. The result is the sun's altitude when the sun is on the prime vertical.

The first part of the stanza is wrong and is attacked by Brahmagupta in his Brahma-Siddhanta, Chapter XI, 22. The sun can cross the prime vertical only when its north declination is less than the latitude of the station.

Symbolically it means :—

$$\text{R sin A} = \frac{\text{Agra} \times \text{R cos } \phi}{\text{R sin } \phi} = \frac{\text{R sin } w \times \text{R sin } l \times \text{R cos } \phi}{\text{R cos } \phi \times \text{R sin } \phi}$$

$$= \frac{\text{R} \times \text{R sin } \delta}{\text{R sin } \phi},$$

which is the correct equation for the altitude of the sun in the prime vertical. For the Indian method *cf.* the *Triprasnadhikara* of Bhaskara's work.

32. The sine of the sun's meridian altitude is called *Parasanku*, and the sine of the meridian zenith distance is the shadow of that *Parasanku*.

33. Multiply the sine of the zenith distance of the culminating point of the ecliptic by the sine of the amplitude of the horoscope (लग्न) *i.e.*, of the point of the ecliptic on the eastern horizon and divide the result by the radius: take the square root of the difference of the square of this result and the square of the sine of the zenith distance of the culminating point of the ecliptic; the result is the sine of the zenith distance of the nonagesimal.

The culminating point of the ecliptic, the nonagesimal and the zenith from a spherical triangle, which is here taken as a plane triangle by Aryabhata as also by the author of the *Surya-Siddhanta*. The first part of the stanza gives correctly the rule for finding the perpendicular from the nonagesimal on the meridian plane.

34. Take the square root of the difference of the squares of the sine of the sun's zenith distance and the sine of the zenith distance of the nonagesimal, the result is the sine of the arc between the sun and the nonagesimal and is called *Dringnati*.

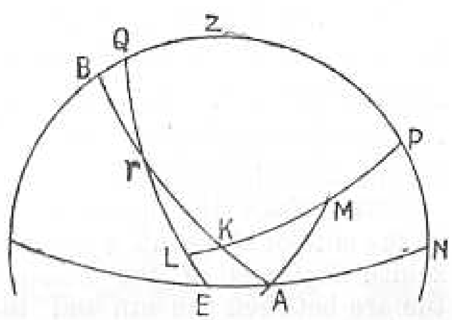
The sun, the nonagesimal and the zenith from a spherical triangle, which is here taken as a plane triangle by Aryabhata. This is found fault with by Brahmagupta in his Brahma-Siddhanta, XI, 27.

The second half of the stanza is of obscure import. In these stanzas Aryabhata aims at giving rules for finding the sun's parallax in longitude. Brahmagupta gives more accurate rules in his work, *vide* B. S. V., 3-5.

35. Multiply the sine of the latitude of the station by the latitude of the moon and divide by the sine of the co-latitude, the result is to be subtracted at the rising and to be added at the setting when the moon is north of the ecliptic and to be respectively added and subtracted when south of the ecliptic.

This is an operation called *Aksha Drik Karma*, one of the two operations known as *Drik Karma*. The word *Drik Karma* means an operation by which is found the point of the ecliptic which rises or sets simultaneously with the moon and other heavenly bodies; it is an operation by which a heavenly body is made fit for being seen simultaneously with a point of the ecliptic. The celestial co-ordinates used being the longitude and latitude, the first step is to convert the co-ordinates to polar longitude and polar latitude which is known by the name *Ayana Drikkarma* which is the *Drikkarma* on the equator. The second step is to allow for the ascensional difference due to the polar latitude and is known as the *Aksha Drikkarma*.

In the figure let PZQB be the meridian, EA the horizon, γA and AM are the longitude and latitude of M. Through P draw PMKL=perpendicular to the equator Eγ Q, then MK is the polar latitude and γK is the polar longitude. When the point K is on the horizon, M having a north polar latitude has already risen. The problem



is first to find the ascensional difference produced by MK and secondly to subtract from γK the part of the ecliptic that rises in this time. To do the first part is to take the difference of the ascensional differences due to the declinations LM and LK, as suggested by Brahma Gupta, X, 18-19 and Bhaskara, Grahaganita, XIII, 6. A rough method of finding the ascensional difference due to MK is this:

When the point L is on the horizon, the angle PEN=PN (where N is the north point)=latitude of the station.

In this position the part of the diurnal circle of K between K and the horizon is thus

$$=\frac{LK\times R\sin\phi}{R\cos\phi}\ \text{approximately}$$

and the corresponding ascensional difference

$$=\frac{LK\times R\sin\phi}{R\cos\phi}\times\frac{R}{R\cos LK}$$

Similarly in the same position, the corresponding part of the diurnal circle of

$$M=\frac{LM\times R\sin\phi}{R\cos\phi}\ \text{approximately,}$$

and the corresponding ascensional difference

$$=\frac{LM\times R\sin\phi}{R\cos\phi}\times\frac{R}{R\cos LM}$$

∴ ascensional difference due to MK

$$=\frac{LM\times R\sin\phi\times R}{R\cos\phi\times R\cos LM}-\frac{LK\times R\sin\phi\times R}{R\cos\phi\times R\cos LK}$$

Now put

$$R=R\cos LM=R\cos LK,$$

∴ the required ascensional difference is

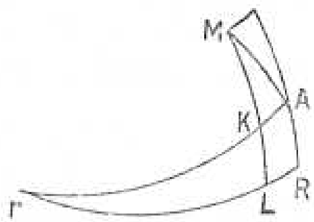
$$\frac{MK\times R\sin\phi}{R\cos\phi}$$

which is a part of the equator; to find the corresponding part of the ecliptic we should divide the result by $\cos\omega\times\sec^2\delta$

(Godfray's Astronomy, p. 159) which is equal to unity according to the rule of Aryabhata. Brahma Gupta objects to this single rule of Aryabhata as it is not applicable when the latitude and the declination are not small. B. S. XI, 34.

36. Multiply versed sine of the moon's longitude by the sine of the extreme declination and by her latitude and divide by the square of the radius; subtract or add the result to the longitude if the moon be in the northerly course, and add or subtract in the southerly course.

The above rule is for finding the polar longitude of the moon, from her given longitude and latitude. In the previous figure it is attempted to find AK and to apply it to γA by this rule



$$AK = \frac{MA \times R \text{ vers } \gamma A \times R \sin \omega}{R^2}$$

If for R vers γA, (R$-$R vers γA) were used the result would have been somewhat accurate. See Brahma Gupta, X, 17. The wrong rule of Aryabhata is severely criticised by him in B. S. XI, 35. Strangely enough this wrong rule is not only accepted but also applied to misinterpret Brahma Gupta by his commentator Chaturveda. Brahma Gupta's rule is improved upon by Bhaskara, see Grahaganita, XIII, 4-5.

37. The moon is water, the sun is fire, the world is earth and the shadow is undoubtedly darkness. The moon covers the sun, the big shadow of the earth, the moon.

38. When the moon being near to her node at the end of the (synodic) month enters the sun or when she enters the shadow of the earth at full moon, it is either a little beyond or before the middle of an eclipse.

39. Muitiply the distance between the earth and the sun by the diameter of the earth and divide by the difference of their diameters, the result is the length of the earth's shadow from the centre of the earth.

40. Multiply the distance between the vertex of the shadow and the moon, by the diameter of the earth and divide by the length of the shadow, the result is the diameter of the shadow at the orbit of the moon.

41. From the square of half the sum of the diameter of the obscured and the obscuring bodies, take the square of the moon's latitude; half the duration of an eclipse is obtained from the square root of the result by using the daily motions of the sun and the moon.

The correction for the parallax of the longitudes and the latitude of the moon in the case of a solar eclipse is not expressly stated. Further in the diagram of the eclipse the moon is taken for the centre as in all Indian astronomical works.

42. Diminish the radius of the shadow by the radius of the moon, from the square of the result subtract the square of the moon's latitude; the square root of the result determines half the time of total obscuration.

43. Subtract the semi-diameter of the moon from the semi-diameter of the shadow, subtract the remainder from the latitude of the moon; the result is that part of the moon which is not obscured.

44. To the square of the latitude add the square of the arc of half duration diminished by the difference of longitudes produced in any given time (from the beginning of the eclipse): square root of the result subtracted from half the sum of the diameters, is the *Grasha*, the part obscured.

45. Sine of the time before or after the noon, multiplied by the sine of the latitude and divided by the radius, is the bearing of the east direction of the equator south of the observer's east on the disc in the eastern half of sphere and north of the observers east in the western half. This is found not only for the middle of the eclipse but also for the instants preceding and following that instant by the half duration. To the above is applied the declination of the longitude of the sun and moon at the time of contact increased by 90° (the algebraic sum represents the bearing of the ecliptic from the east of the centre of the disc at that instant).

Aryabhata here speaks of the two *valanas,* the second of which is called the position angle, and his rule is almost the same as that given in the modern Surya-Siddhanta. For a

complete discussion *cf.*, the *valanas* of the Goladhayaya VIII 30-74 and also the commentary thereon. See also my Papers.

46. At the commencement of an eclipse, the moon becomes of a smoky colour, appears dark in a partial eclipse, becomes brown in a total obscuration also dark brown when entirely inside the shadow (*i.e.*, in a total eclipse).

47. At the contact of the circumferences of the sun and the moon, one eighth of the disc of the sun though eclipsed is not seen to be so; this is due to the extreme brightness of the sun and to the moon's circumference being also transparent.

48. For this work the sun has been determined from the conjunction of the earth (*i.e.*, the horizon) and the sun, and the moon from the conjunction of the sun and the moon; similarly all the "star planets" have been found from their conjunctions with the moon.

Aryabhata here gives us the method by which he determined the mean notions of planets. The sun, he determined by the conjunction of the horizon and the sun. The year which he gives is no doubt the sidereal year. The changing appearance of the heavens in different seasons must have led to the determination of the year by the heliacal rising of a bright star *i.e.*, by observing the time that elapses between the rising of a certain bright star and the sun at intervals of 365 and 366 days. Such a year is necessarily the sidereal year. It is needless to say that such observations are best made in winter.

The moon he determined from the conjunction of the sun and moon, *i.e.*, from the synodic month. When the year is known the sidereal month can be deduced from the equation

$$\frac{1}{\text{synodic month}}=\frac{1}{\text{sidereal month}}-\frac{1}{\text{sidereal year}}.$$

This synodic month may have been determined from the mean period of the heliacal rising of the moon, or from the eclipses of the sun separated by a considerable number of years. Now to come to the other half of the stanza.

Aryabhata in the second half of the stanza says that he determined the mean notions of the star planets from their conjunctions with the moon. If we suppose the orbits of the planets to be coplanar, circular and concentric about the sun, and to be described uniformly, in the case of a superior planet starting from an instant of conjunction with the sun, the length of the rth period of conjunction is obtained in the form,

$$P+A_1\text{Cos}\,\frac{\pi\{(2r+1)P-2k\}}{P'}-B_1\text{Cos}\,\frac{2\pi\{(2r+1)P-2k\}}{P'}$$
$$+C_1\text{Cos}\,\frac{3\pi\{(2r+1)P-2k\}}{P'}$$

where the first fractional period bringing in the constant k is not counted. P denotes the mean period of conjunction with the moon obtained by assuming the superior planet and the moon to be moving uniformly round the earth, P′ is the synodic period of the superior planet, A_1, B_1, C_1, are the constants depending on the ratio of the mean distance of the earth to the mean distance of the superior planet and on P and P′.

Now let us suppose that this length of period is repeated after a complete cycle of the harmonic terms in the r'th period then

$$\frac{\pi\{(2r+1)P-2k\}}{P'}+2q\pi=\frac{\pi\{(2r'+1)P-2k\}}{P'}$$

where q is the least suitable integer.

$$\therefore\frac{P'}{P}=\frac{r'-r}{q}$$

Here $r'-r$ denotes the number of periods of conjuction observed between the $(r+1)$th and the $(r'+1)$th conjuction. For Mars P′ is about 779·933 days; P=28·453 the first four convergents to

$$\frac{P'}{P}\text{ are }\frac{27}{1},\frac{55}{2},\frac{137}{5},\frac{329}{12}.$$

Let us take the convergent $\frac{137}{5}$ for Mars, *i.e.*, take 137 periods of conjunction with the moon = 5 synodic periods of Mars. There was a conjunction of Mars with the moon on Jan. 8, 10h. 7m. G. M. T. in 1908 and the 137th conjunction fell on September 10, 8h. 45 m. G. M. T. in 1918. The number of days elapsed in these 137 periods was 3897·94305. The moon's motion in 3897·94305 days = 142·6685975 revolutions, the mean sidereal month being taken at 27·3216661 days. On deducting 137, the number of periods of conjunction with the moon, Mars' motion was 5·6685975 revolutions in 3897·94305 days, whence the deduced sidereal period of Mars became 687·62 days nearly and according to Aryabhata it is 686·99974 days, while the actual period is 686·9797 days.

In the case of inferior planets a similar treatment shows that the mean period of conjuction with the moon is equal to the synodic month, *i.e.*, the same as that of the conjunction with the sun. Hence the mean period of revolution of an inferior planet round the earth is equal to the sidereal year. It thus appears that the method of Aryabhata has a sound theory to back it.

49. Thus from the ocean of true and eroneous knowledge has been raised by me with the help of the boat of my own intelligence, the precious jewel of true knowledge which so long lay sunk therein.

50. The same science which was always true and went by the name of *Shvayambhuva* (Brahma-Siddhanta) is now published under the name of *Aryabhatiyam*. Loss of religious merit and longevity will be the share of the person who will vilify it by malicious criticisms.

The End of Section IV, named the Gola.

NOTE ON THE LIBRARY OF ABDUR RAHIM KHAN KHANAN, THE FIRST PRIME MINISTER OF THE EMPEROR AKBAR

BY

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Islamic history teems with illustrations of patronage of learning and erudition. In energy, clime and country, persons of light and leading, professing the Islamic faith, have endeavoured to preserve the torch of learning from extinction. Learning has always been one of the main planks in the platform of the Islamic faith, and may be likened to a centrifugal force for religious propaganda. Learning flourished *pari passu* with religion, a statement applicable with particular cogency to Islam. From the very dawn of the faith, learning travelled from the deserts of Madina, and took an eccentric but firm course through many lands shedding its lustre, specially in Europe, then in gross literary darkness. By degrees the torch of Islamic learning was carried to India where it consumed in its flames much of the prevailing sophistry. From the Hindu Kush to Cape Comorin learning, based on Islamic principles and under Islamic patronage, spread with great vigour, and learned men, encouraged by kings and potentates, founded schools, colleges and libraries. History teems with names of Ulamas who flourished in different periods of Islamic Sovereignty in India.

Here we are concerned with the Moghal period and specially with the reign of Akbar the Great, in whose time